Large gaps between consecutive zeros, on the critical line, of the Riemann zeta-function

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Abstract. We show that for any sufficiently large T, there exists a subinterval of $[T, 2T]$ of length at least $2.766 \times \frac{2\pi}{\log T}$, in which the function $t \mapsto \zeta(\frac{1}{2} + it)$ has no zeros.

1 Introduction

It is well-known (see for example [13] or [2]) that the Riemann zeta-function $\zeta(s)$ has so called trivial zeros at $s = -2, -4, -6, \ldots$ and that all the other zeros $s = \sigma + it$ (the non-trivial ones) lie in the critical strip, i.e. satisfy $0 < \sigma < 1$. The Riemann Hypothesis (RH) is a conjecture saying that in fact all the non-trivial zeros must lie on the critical line, i.e. satisfy $\sigma = 1/2$. Levinson [10] showed that at least a third of the non-trivial zeros of the Riemann zeta-function lie on the critical line. However, to this day we do not know the whole truth about the horizontal distribution of the zeros.

It is also well-known that the number of non-trivial zeros with ordinates in $[0, T]$ is $\frac{T \log T}{2\pi} + O(T)$, which tells us that the average difference of the ordinates of two consecutive zeros at height T is approximately $2\pi / \log T$. Denote by $\{\gamma_n\}$ the sequence of ordinates of all zeros of $\zeta(s)$ in the upper halfplane, ordered in non-decreasing order. A natural question to ask is what one can say about

$$\mu := \liminf_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{(2\pi/\log \gamma_n)} \quad \text{and } \lambda := \limsup_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{(2\pi/\log \gamma_n)}.$$

In 1946, Selberg [12] remarked that $\mu < 1$ and $\lambda > 1$. Although this is still the only known unconditional result, it is believed to be far from the whole truth. Indeed, in 1973, Montgomery [11] predicted that $\mu = 0$ and $\lambda = \infty$. On the assumption of RH, Feng and Wu [4] recently obtained $\mu \leqslant 0.5154$ and $\lambda \geqslant 2.7327$. In this article only large gaps are considered and our overall strategy was first used by Hall [6]. We show the following:

Theorem 1 (Main Theorem)**.** *For any sufficiently large T, there exists a subinterval of $[T, 2T]$ of length at least $2.766 \times \frac{2\pi}{\log T}$, in which the function $t \mapsto \zeta(\frac{1}{2} + it)$ has no zeros.*

Remark 1. Notice that if we assume RH, then Theorem 1 implies that $\lambda \geqslant 2.766$.

2 Building-stones in the proof of Theorem 1

2.1 Introducing the function $P(t, u, v, \kappa)$

Definition 1. Define

$$P(t, u, v, \kappa) := \exp(vi\theta(t)) M(\tfrac{1}{2} + it)\zeta(\tfrac{1}{2} + it)\zeta(\tfrac{1}{2} + it + \tfrac{i\kappa}{\log T}), \tag{1}$$

where

$$\theta(t) := \mathrm{Im}\Big(\log\Big(\Gamma\Big(\frac{1}{4} + \frac{it}{2}\Big)\Big)\Big) - \frac{t}{2}\log\pi \tag{2}$$

and

$$M(s) := \sum_{h \leqslant T^u} \frac{1}{h^s}, \tag{3}$$

with $0 < u < 1/11$.

2.2 Main Assumption

We will now make an "assumption".

Main Assumption: Suppose that all the gaps between consecutive zeros of the function $t \mapsto P(t, u, v, \kappa)$ with $t \in [T, 2T - \frac{\kappa}{\log T}]$ are[1] at most $\frac{\kappa}{\log T}$.

Remark 2. For a suitable choice of κ, u and v, we will eventually prove that our Main Assumption leads to a contradiction.

2.3 Immediate consequences of our Main Assumption

Denote the zeros of $P(t, u, v, \kappa)$ with $T \leqslant t \leqslant 2T - \frac{\kappa}{\log T}$ by $t_1, t_2, ..., t_N$, ordered in non-decreasing order. Our Main Assumption implies that

$$t_{i+1} - t_i \leqslant \frac{\kappa}{\log T}, \tag{4}$$

for $i = 1, 2, ..., N - 1$.

Remark 3. For future need we note here that our Main Assumption implies that

$$t_1 \leqslant T + \frac{\kappa}{\log T} \text{ and } t_N \geqslant 2T - \frac{2\kappa}{\log T}.$$

[1]For either of the two zeros near the endpoints of the interval, we will here mean the distance from them to the respective endpoint.

2.4 Wirtinger's inequality and an application of it

We begin with the statement of the simplest version of Wirtinger's inequality.

Theorem 2. *Suppose that $y(t)$ is a continuously differentiable function which satisfies $y(0) = y(\pi) = 0$. Then*

$$\int_0^\pi |y(t)|^2\, dt \leqslant \int_0^\pi |y'(t)|^2\, dt. \tag{5}$$

Proof. For the case when $y(t)$ is a real-valued function, the reader is referred to Theorem 256 in Hardy, Littlewood and Pólya's [8].

Say now that $y(t) = y_1(t) + iy_2(t)$, with y_1 and y_2 thus being real-valued continuously differentiable functions. Then clearly $y'(t) = y_1'(t) + iy_2'(t)$. What we want to show is

$$\int_0^\pi y_1(t)^2 + y_2(t)^2\, dt \leqslant \int_0^\pi y_1'(t)^2 + y_2'(t)^2\, dt,$$

but this immediately follows from the known (real) case. □

Corollary 1. *For $i = 1, 2, ..., N-1$ we have*

$$\begin{aligned}\int_{t_i}^{t_{i+1}} |P(t,u,v,\kappa)|^2\, dt &\leqslant \Big(\frac{t_{i+1}-t_i}{\pi}\Big)^2 \int_{t_i}^{t_{i+1}} |P'(t,u,v,\kappa)|^2\, dt \\ &\leqslant \Big(\frac{\kappa}{\pi\log T}\Big)^2 \int_{t_i}^{t_{i+1}} |P'(t,u,v,\kappa)|^2\, dt.\end{aligned}$$

Proof. One may make a linear substitution in Theorem 2 to obtain a similar result if the function $y(t)$ has zeros at two general points a and b. We do so for the function $P(t,u,v,\kappa)$, which is continuously differentiable, and this gives us the first inequality. The latter inequality follows immediately from (4). □

Simply summing up the inequalities in Corollary 1 for $i = 1, 2, ..., N-1$, we obtain

$$\int_{t_1}^{t_N} |P(t,u,v,\kappa)|^2\, dt \leqslant \Big(\frac{\kappa}{\pi\log T}\Big)^2 \int_{t_1}^{t_N} |P'(t,u,v,\kappa)|^2\, dt. \tag{6}$$

2.5 Choosing weight-functions

Definition 2. With $\eta > 0$ being any suitably small (fixed) constant, we define

$$h(x) := \begin{cases} \exp(-\eta T_0/x) & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0, \end{cases} \tag{7}$$

with

$$T_0 = T^{1-\epsilon}. \tag{8}$$

Then clearly $h(x)$ is C^∞ and $h(x) \leqslant 1$. Also,

$$h'(x) := \begin{cases} \eta T_0^{-1}(T_0/x)^2 \exp(-\eta T_0/x) & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0, \end{cases}$$

which is seen to imply $h'(x) \ll T_0^{-1}$. And more generally one finds that $h^{(j)}(x) \ll_j T_0^{-j}$.

Now take

$$w_-(x) = h(x - T - T_0)h(2T - T_0 - x) \tag{9}$$

and

$$w_+(x) = \exp(2\eta)h(x - T + T_0)h(2T + T_0 - x). \tag{10}$$

Remembering Remark 3, it is easily seen that (6) implies

Corollary 2.

$$\int_{-\infty}^{\infty} w_-(t)|P(t,u,v,\kappa)|^2 \, dt \leqslant \left(\frac{\kappa}{\pi \log T}\right)^2 \int_{-\infty}^{\infty} w_+(t)|P'(t,u,v,\kappa)|^2 \, dt. \tag{11}$$

3 Going from Corollary 2 to Theorem 1

In this section we will write down asymptotic estimates for the Left Hand Side (LHS) and the Right Hand Side (RHS) in (11), and use these to obtain an inequality in terms of κ.

3.1 Giving names to some integrals

Recall that $P(t,u,v,\kappa) = \exp(vi\theta(t))M(\frac{1}{2}+it)\zeta(\frac{1}{2}+it)\zeta(\frac{1}{2}+it+\frac{i\kappa}{\log T})$. Obviously

$$|P(t,u,v,\kappa)|^2 = |M(\tfrac{1}{2}+it)|^2|\zeta(\tfrac{1}{2}+it)|^2|\zeta(\tfrac{1}{2}+it+\tfrac{i\kappa}{\log T})|^2. \tag{12}$$

Next, using

$$M'(s) = -\sum_{h \leqslant T^u} \frac{\log h}{h^s} = N(s) - \log(T^u)M(s),$$

where

$$N(s) := \sum_{h \leqslant T^u} \frac{\log(T^u/h)}{h^s}, \tag{13}$$

we find that

$$\begin{aligned}\frac{P'(t,u,v,\kappa)}{i\exp(vi\theta(t))} &= \big\{v\theta'(t) - \log(T^u)\big\} M(\tfrac12+it)\zeta(\tfrac12+it)\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})\\ &\quad + N(\tfrac12+it)\zeta(\tfrac12+it)\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})\\ &\quad + M(\tfrac12+it)\zeta'(\tfrac12+it)\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})\\ &\quad + M(\tfrac12+it)\zeta(\tfrac12+it)\zeta'(\tfrac12+it+\tfrac{i\kappa}{\log T}).\end{aligned}$$

Thus

$$\begin{aligned}&|P'(t,u,v,\kappa)|^2 = \big\{v\theta'(t) - \log(T^u)\big\}^2 |M(\tfrac12+it)|^2|\zeta(\tfrac12+it)|^2|\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2 \\ &+ 2\big\{v\theta'(t) - \log(T^u)\big\}\mathrm{Re}\Big[M(\tfrac12+it)N(\tfrac12-it)|\zeta(\tfrac12+it)|^2|\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2\Big]\\ &+ |N(\tfrac12+it)|^2|\zeta(\tfrac12+it)|^2|\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2\\ &+ 2\big\{v\theta'(t) - \log(T^u)\big\}\mathrm{Re}\Big[|M(\tfrac12+it)|^2\zeta'(\tfrac12+it)\zeta(\tfrac12-it)|\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2\Big]\\ &+ 2\big\{v\theta'(t) - \log(T^u)\big\}\mathrm{Re}\Big[|M(\tfrac12+it)|^2|\zeta(\tfrac12+it)|^2\zeta'(\tfrac12+it+\tfrac{i\kappa}{\log T})\zeta(\tfrac12-it-\tfrac{i\kappa}{\log T})\Big]\\ &+ |M(\tfrac12+it)|^2|\zeta'(\tfrac12+it)|^2|\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2\\ &+ |M(\tfrac12+it)|^2|\zeta(\tfrac12+it)|^2|\zeta'(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2\\ &+ 2\mathrm{Re}\Big[|M(\tfrac12+it)|^2\zeta'(\tfrac12+it)\zeta(\tfrac12-it)\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})\zeta'(\tfrac12-it-\tfrac{i\kappa}{\log T})\Big]\\ &+ 2\mathrm{Re}\Big[M(\tfrac12+it)N(\tfrac12-it)\zeta'(\tfrac12+it)\zeta(\tfrac12-it)|\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2\Big]\\ &+ 2\mathrm{Re}\Big[M(\tfrac12+it)N(\tfrac12-it)|\zeta(\tfrac12+it)|^2\zeta'(\tfrac12+it+\tfrac{i\kappa}{\log T})\zeta(\tfrac12-it-\tfrac{i\kappa}{\log T})\Big].\end{aligned} \tag{14}$$

For future need we now introduce some notation. Let

$$A_\pm = \int_{-\infty}^{\infty} w_\pm(t)|M(\tfrac12+it)|^2|\zeta(\tfrac12+it)|^2|\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2\,dt, \tag{15}$$

$$B_\pm = \int_{-\infty}^{\infty} w_\pm(t)\mathrm{Re}\Big[M(\tfrac12+it)N(\tfrac12-it)|\zeta(\tfrac12+it)|^2|\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2\Big]\,dt, \tag{16}$$

$$C_\pm = \int_{-\infty}^{\infty} w_\pm(t)|N(\tfrac12+it)|^2|\zeta(\tfrac12+it)|^2|\zeta(\tfrac12+it+\tfrac{i\kappa}{\log T})|^2\,dt, \tag{17}$$

$$D_\pm = \int_{-\infty}^{\infty} w_\pm(t)\mathrm{Re}\Big[|M(\tfrac{1}{2}+it)|^2\zeta'(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}-it)|\zeta(\tfrac{1}{2}+it+\tfrac{i\kappa}{\log T})|^2\Big]\,dt, \tag{18}$$

$$E_\pm = \int_{-\infty}^{\infty} w_\pm(t)\mathrm{Re}\Big[|M(\tfrac{1}{2}+it)|^2|\zeta(\tfrac{1}{2}+it)|^2\zeta'(\tfrac{1}{2}+it+\tfrac{i\kappa}{\log T})\zeta(\tfrac{1}{2}-it-\tfrac{i\kappa}{\log T})\Big]\,dt, \tag{19}$$

$$F_\pm = \int_{-\infty}^{\infty} w_\pm(t)|M(\tfrac{1}{2}+it)|^2|\zeta'(\tfrac{1}{2}+it)|^2|\zeta(\tfrac{1}{2}+it+\tfrac{i\kappa}{\log T})|^2\,dt, \tag{20}$$

$$G_\pm = \int_{-\infty}^{\infty} w_\pm(t)|M(\tfrac{1}{2}+it)|^2|\zeta(\tfrac{1}{2}+it)|^2|\zeta'(\tfrac{1}{2}+it+\tfrac{i\kappa}{\log T})|^2\,dt, \tag{21}$$

$$H_\pm = \int_{-\infty}^{\infty} w_\pm(t)\mathrm{Re}\Big[|M(\tfrac{1}{2}+it)|^2\zeta'(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}-it)\zeta(\tfrac{1}{2}+it+\tfrac{i\kappa}{\log T})\zeta'(\tfrac{1}{2}-it-\tfrac{i\kappa}{\log T})\Big]\,dt, \tag{22}$$

$$I_\pm = \int_{-\infty}^{\infty} w_\pm(t)\mathrm{Re}\Big[M(\tfrac{1}{2}+it)N(\tfrac{1}{2}-it)\zeta'(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}-it)|\zeta(\tfrac{1}{2}+it+\tfrac{i\kappa}{\log T})|^2\Big]\,dt \tag{23}$$

and

$$J_\pm = \int_{-\infty}^{\infty} w_\pm(t)\mathrm{Re}\Big[M(\tfrac{1}{2}+it)N(\tfrac{1}{2}-it)|\zeta(\tfrac{1}{2}+it)|^2\zeta'(\tfrac{1}{2}+it+\tfrac{i\kappa}{\log T})\zeta(\tfrac{1}{2}-it-\tfrac{i\kappa}{\log T})\Big]\,dt. \tag{24}$$

3.2 Evaluation of the integrals defined in Section 3.1

The reader is referred to Section 4 for details on how to evaluate (asymptotically) the integrals (15)-(24). Below we give the answers.

$$A_\pm = A_\kappa \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_\pm(t)\,dt\Big] \cdot \log^9 T + O(T\log^8 T), \tag{25}$$

$$B_\pm = B_\kappa \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_\pm(t)\,dt\Big] \cdot \log^{10} T + O(T\log^9 T), \tag{26}$$

$$C_\pm = C_\kappa \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_\pm(t)\,dt\Big] \cdot \log^{11} T + O(T\log^{10} T), \tag{27}$$

$$D_\pm = D_\kappa \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_\pm(t)\,dt\Big] \cdot \log^{10} T + O(T\log^9 T), \tag{28}$$

$$E_\pm = E_\kappa \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_\pm(t)\,dt\Big] \cdot \log^{10} T + O(T\log^9 T), \tag{29}$$

$$F_{\pm} = F_{\kappa} \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_{\pm}(t)\ dt \Big] \cdot \log^{11} T + O(T \log^{10} T), \tag{30}$$

$$G_{\pm} = G_{\kappa} \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_{\pm}(t)\ dt \Big] \cdot \log^{11} T + O(T \log^{10} T), \tag{31}$$

$$H_{\pm} = H_{\kappa} \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_{\pm}(t)\ dt \Big] \cdot \log^{11} T + O(T \log^{10} T), \tag{32}$$

$$I_{\pm} = I_{\kappa} \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_{\pm}(t)\ dt \Big] \cdot \log^{11} T + O(T \log^{10} T) \tag{33}$$

and

$$J_{\pm} = J_{\kappa} \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_{\pm}(t)\ dt \Big] \cdot \log^{11} T + O(T \log^{10} T), \tag{34}$$

with

$$a_3 = \prod_p \Big\{ \Big(1 + \frac{4}{p} + \frac{1}{p^2}\Big)\Big(1 - \frac{1}{p}\Big)^4 \Big\} \tag{35}$$

and where the constants are given by

$$\begin{aligned} A_{\kappa} = {} & \frac{(-10)}{\kappa^8} + \frac{(2u - 2u^2 + \frac{u^3}{3})}{\kappa^6} + \frac{(\frac{u^3}{3} - \frac{u^4}{4})}{\kappa^4} + \frac{8\sin(\kappa u)}{\kappa^9} + \frac{(10 - 8u)\cos(\kappa u)}{\kappa^8} \\ & + \frac{(-4 + 10u - 4u^2)\sin(\kappa u)}{\kappa^7} + \frac{(2u - 3u^2 + u^3)\cos(\kappa u)}{\kappa^6} + \frac{(-8)\sin\kappa}{\kappa^9} + \frac{(\frac{-u^3}{3})\cos\kappa}{\kappa^6} \\ & + \frac{8\sin(\kappa(1-u))}{\kappa^9} + \frac{(8u)\cos(\kappa(1-u))}{\kappa^8} + \frac{(-4u^2)\sin(\kappa(1-u))}{\kappa^7} + \frac{(-u^3)\cos(\kappa(1-u))}{\kappa^6}, \end{aligned} \tag{36}$$

$$\begin{aligned} B_{\kappa} = {} & \frac{(1 - 2u)}{\kappa^8} + \frac{(-\frac{u^3}{3} + \frac{u^4}{6})}{\kappa^6} + \frac{(\frac{u^4}{8} - \frac{u^5}{12})}{\kappa^4} + \frac{(-1 + 2u)\cos(\kappa u)}{\kappa^8} \\ & + \frac{(-u + 2u^2 - \frac{u^3}{3})\sin(\kappa u)}{\kappa^7} + \frac{(\frac{u^2}{2} - \frac{2u^3}{3} + \frac{u^4}{6})\cos(\kappa u)}{\kappa^6} + \frac{(\frac{u^3}{3})\sin\kappa}{\kappa^7} + \frac{(\frac{-u^4}{6})\cos\kappa}{\kappa^6} \\ & + \frac{(-\frac{u^3}{3})\sin(\kappa(1-u))}{\kappa^7} + \frac{(-\frac{u^4}{6})\cos(\kappa(1-u))}{\kappa^6}, \end{aligned} \tag{37}$$

$$\begin{aligned} C_{\kappa} = {} & \frac{(-20)}{\kappa^{10}} + \frac{(2u - 2u^2)}{\kappa^8} + \frac{(-\frac{u^4}{6} + \frac{u^5}{15})}{\kappa^6} + \frac{(\frac{u^5}{20} - \frac{u^6}{36})}{\kappa^4} + \frac{12\sin(\kappa u)}{\kappa^{11}} \\ & + \frac{(20 - 12u)\cos(\kappa u)}{\kappa^{10}} + \frac{(-6 + 20u - 6u^2)\sin(\kappa u)}{\kappa^9} + \frac{(4u - 8u^2 + 2u^3)\cos(\kappa u)}{\kappa^8} \\ & + \frac{(u^2 - \frac{4u^3}{3} + \frac{u^4}{3})\sin(\kappa u)}{\kappa^7} + \frac{(-12)\sin\kappa}{\kappa^{11}} + \frac{(\frac{u^4}{6})\sin\kappa}{\kappa^7} + \frac{(-\frac{u^5}{15})\cos\kappa}{\kappa^6} \\ & + \frac{12\sin(\kappa(1-u))}{\kappa^{11}} + \frac{(12u)\cos(\kappa(1-u))}{\kappa^{10}} + \frac{(-6u^2)\sin(\kappa(1-u))}{\kappa^9} \\ & + \frac{(-2u^3)\cos(\kappa(1-u))}{\kappa^8} + \frac{(\frac{u^4}{3})\sin(\kappa(1-u))}{\kappa^7}, \end{aligned} \tag{38}$$

$$D_\kappa = -\frac{A_\kappa}{2}, \tag{39}$$

$$E_\kappa = -\frac{A_\kappa}{2}, \tag{40}$$

$$\begin{aligned}
F_\kappa = {} & \frac{(-66)}{\kappa^{10}} + \frac{(-\frac{11}{3} + 8u - 8u^2 + \frac{4u^3}{3})}{\kappa^8} + \frac{(\frac{2u}{3} - \frac{2u^2}{3} + \frac{5u^3}{6} - \frac{2u^4}{3} - \frac{u^5}{60})}{\kappa^6} \\
& + \frac{(\frac{u^3}{9} - \frac{u^4}{8} + \frac{u^5}{20} - \frac{u^6}{72})}{\kappa^4} + \frac{84\sin(\kappa u)}{\kappa^{11}} + \frac{(66 - 84u)\cos(\kappa u)}{\kappa^{10}} + \frac{(-16 + 66u - 42u^2)\sin(\kappa u)}{\kappa^9} \\
& + \frac{(\frac{11}{3} + 8u - 25u^2 + \frac{38u^3}{3})\cos(\kappa u)}{\kappa^8} + \frac{(-\frac{4}{3} + \frac{11u}{3} - \frac{11u^3}{3} + \frac{11u^4}{6})\sin(\kappa u)}{\kappa^7} \\
& + \frac{(\frac{2u}{3} - \frac{7u^2}{6} + \frac{u^3}{2} + \frac{u^4}{12} - \frac{u^5}{12})\cos(\kappa u)}{\kappa^6} + \frac{(-84)\sin\kappa}{\kappa^{11}} + \frac{26\cos\kappa}{\kappa^{10}} + \frac{(-\frac{4u^3}{3})\cos\kappa}{\kappa^8} \\
& + \frac{(-\frac{2u^3}{3} + \frac{u^4}{3})\sin\kappa}{\kappa^7} + \frac{(-\frac{u^4}{12} + \frac{u^5}{60})\cos\kappa}{\kappa^6} + \frac{84\sin(\kappa(1-u))}{\kappa^{11}} + \frac{(-26 + 84u)\cos(\kappa(1-u))}{\kappa^{10}} \\
& + \frac{(26u - 42u^2)\sin(\kappa(1-u))}{\kappa^9} + \frac{(13u^2 - \frac{38u^3}{3})\cos(\kappa(1-u))}{\kappa^8} \\
& + \frac{(-\frac{11u^3}{3} + \frac{11u^4}{6})\sin(\kappa(1-u))}{\kappa^7} + \frac{(-\frac{u^4}{3} + \frac{u^5}{12})\cos(\kappa(1-u))}{\kappa^6},
\end{aligned} \tag{41}$$

$$\begin{aligned}
G_\kappa = {} & \frac{(-148)}{\kappa^{10}} + \frac{(-\frac{14}{3} + 18u - 18u^2 + 3u^3)}{\kappa^8} + \frac{(\frac{2u}{3} - u^2 + \frac{11u^3}{6} - \frac{7u^4}{6})}{\kappa^6} + \frac{(\frac{u^3}{9} - \frac{u^4}{12})}{\kappa^4} \\
& + \frac{152\sin(\kappa u)}{\kappa^{11}} + \frac{(148 - 152u)\cos(\kappa u)}{\kappa^{10}} + \frac{(-40 + 148u - 76u^2)\sin(\kappa u)}{\kappa^9} \\
& + \frac{(\frac{14}{3} + 22u - 56u^2 + \frac{67u^3}{3})\cos(\kappa u)}{\kappa^8} + \frac{(-\frac{4}{3} + \frac{14u}{3} + 2u^2 - \frac{26u^3}{3} + \frac{10u^4}{3})\sin(\kappa u)}{\kappa^7} \\
& + \frac{(\frac{2u}{3} - \frac{4u^2}{3} + \frac{u^3}{2} + \frac{u^4}{3} - \frac{u^5}{6})\cos(\kappa u)}{\kappa^6} + \frac{(-152)\sin\kappa}{\kappa^{11}} + \frac{36\cos\kappa}{\kappa^{10}} + \frac{(-3u^3)\cos\kappa}{\kappa^8} \\
& + \frac{(-u^3)\sin\kappa}{\kappa^7} + \frac{152\sin(\kappa(1-u))}{\kappa^{11}} + \frac{(-36 + 152u)\cos(\kappa(1-u))}{\kappa^{10}} \\
& + \frac{(36u - 76u^2)\sin(\kappa(1-u))}{\kappa^9} + \frac{(18u^2 - \frac{67u^3}{3})\cos(\kappa(1-u))}{\kappa^8} \\
& + \frac{(-5u^3 + \frac{10u^4}{3})\sin(\kappa(1-u))}{\kappa^7} + \frac{(-\frac{u^4}{2} + \frac{u^5}{6})\cos(\kappa(1-u))}{\kappa^6},
\end{aligned} \tag{42}$$

$$
\begin{aligned}
H_\kappa = {} & \frac{117}{\kappa^{10}} + \frac{(-\frac{5}{2} - 14u + 14u^2 - \frac{7u^3}{3})}{\kappa^8} + \frac{(\frac{u}{2} - \frac{u^2}{2} - \frac{7u^3}{6} + \frac{25u^4}{24})}{\kappa^6} + \frac{(\frac{u^3}{12} - \frac{u^4}{16})}{\kappa^4} \\
& + \frac{(-130)\sin(\kappa u)}{\kappa^{11}} + \frac{(-117 + 130u)\cos(\kappa u)}{\kappa^{10}} + \frac{(38 - 117u + 65u^2)\sin(\kappa u)}{\kappa^9} \\
& + \frac{(\frac{5}{2} - 24u + \frac{89u^2}{2} - \frac{58u^3}{3})\cos(\kappa u)}{\kappa^8} + \frac{(-1 + \frac{5u}{2} - 5u^2 + 7u^3 - \frac{17u^4}{6})\sin(\kappa u)}{\kappa^7} \\
& + \frac{(\frac{u}{2} - \frac{3u^2}{4} + \frac{u^3}{2} - \frac{5u^4}{12} + \frac{u^5}{6})\cos(\kappa u)}{\kappa^6} + \frac{130\sin\kappa}{\kappa^{11}} + \frac{(-31)\cos\kappa}{\kappa^{10}} + \frac{(-4)\sin\kappa}{\kappa^9} \\
& + \frac{(\frac{7u^3}{3})\cos\kappa}{\kappa^8} + \frac{(\frac{5u^3}{6} - \frac{u^4}{4})\sin\kappa}{\kappa^7} + \frac{(-\frac{u^3}{6} + \frac{u^4}{24})\cos\kappa}{\kappa^6} + \frac{(-130)\sin(\kappa(1-u))}{\kappa^{11}} \\
& + \frac{(31 - 130u)\cos(\kappa(1-u))}{\kappa^{10}} + \frac{(4 - 31u + 65u^2)\sin(\kappa(1-u))}{\kappa^9} \\
& + \frac{(4u - \frac{31u^2}{2} + \frac{58u^3}{3})\cos(\kappa(1-u))}{\kappa^8} + \frac{(-2u^2 + \frac{13u^3}{3} - \frac{17u^4}{6})\sin(\kappa(1-u))}{\kappa^7} \\
& + \frac{(-\frac{u^3}{2} + \frac{5u^4}{12} - \frac{u^5}{6})\cos(\kappa(1-u))}{\kappa^6},
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
I_\kappa = {} & \frac{(-35)}{\kappa^{10}} + \frac{(-\frac{1}{2} + 5u - 4u^2 + \frac{2u^3}{3})}{\kappa^8} + \frac{(\frac{5u^3}{12} - \frac{u^4}{4} + \frac{u^5}{40})}{\kappa^6} + \frac{(-\frac{u^4}{16} + \frac{u^5}{20} - \frac{u^6}{144})}{\kappa^4} \\
& + \frac{32\sin(\kappa u)}{\kappa^{11}} + \frac{(35 - 32u)\cos(\kappa u)}{\kappa^{10}} + \frac{(-11 + 35u - 16u^2)\sin(\kappa u)}{\kappa^9} \\
& + \frac{(\frac{1}{2} + 6u - \frac{27u^2}{2} + \frac{14u^3}{3})\cos(\kappa u)}{\kappa^8} + \frac{(\frac{u}{2} + \frac{u^2}{2} - \frac{13u^3}{6} + \frac{3u^4}{4})\sin(\kappa u)}{\kappa^7} \\
& + \frac{(-\frac{u^2}{4} + \frac{u^3}{4} + \frac{u^4}{24} - \frac{u^5}{24})\cos(\kappa u)}{\kappa^6} + \frac{(-32)\sin\kappa}{\kappa^{11}} + \frac{5\cos\kappa}{\kappa^{10}} + \frac{(-\frac{2u^3}{3})\cos\kappa}{\kappa^8} \\
& + \frac{(-\frac{u^3}{3} - \frac{u^4}{12})\sin\kappa}{\kappa^7} + \frac{(\frac{u^4}{8} - \frac{u^5}{40})\cos\kappa}{\kappa^6} + \frac{32\sin(\kappa(1-u))}{\kappa^{11}} + \frac{(-5 + 32u)\cos(\kappa(1-u))}{\kappa^{10}} \\
& + \frac{(5u - 16u^2)\sin(\kappa(1-u))}{\kappa^9} + \frac{(\frac{5u^2}{2} - \frac{14u^3}{3})\cos(\kappa(1-u))}{\kappa^8} + \frac{(-\frac{u^3}{2} + \frac{3u^4}{4})\sin(\kappa(1-u))}{\kappa^7} \\
& + \frac{(\frac{u^5}{24})\cos(\kappa(1-u))}{\kappa^6}
\end{aligned} \tag{44}
$$

and finally

$$
\begin{aligned}
J_\kappa = {} & \frac{63}{\kappa^{10}} + \frac{(-\frac{1}{2} - 6u + 7u^2 - \frac{7u^3}{6})}{\kappa^8} + \frac{(-\frac{u^3}{4} + \frac{u^4}{8})}{\kappa^6} + \frac{(-\frac{u^4}{16} + \frac{u^5}{24})}{\kappa^4} + \frac{(-50)\sin(\kappa u)}{\kappa^{11}} \qquad (45)\\
& + \frac{(-63 + 50u)\cos(\kappa u)}{\kappa^{10}} + \frac{(20 - 63u + 25u^2)\sin(\kappa u)}{\kappa^9} + \frac{(\frac{1}{2} - 14u + \frac{49u^2}{2} - \frac{43u^3}{6})\cos(\kappa u)}{\kappa^8}\\
& + \frac{(\frac{u}{2} - 4u^2 + \frac{14u^3}{3} - \frac{7u^4}{6})\sin(\kappa u)}{\kappa^7} + \frac{(-\frac{u^2}{4} + \frac{7u^3}{12} - \frac{5u^4}{12} + \frac{u^5}{12})\cos(\kappa u)}{\kappa^6} + \frac{50\sin\kappa}{\kappa^{11}}\\
& + \frac{(-5)\cos\kappa}{\kappa^{10}} + \frac{(\frac{7u^3}{6})\cos\kappa}{\kappa^8} + \frac{(\frac{u^4}{4})\sin\kappa}{\kappa^7} + \frac{(\frac{u^4}{24})\cos\kappa}{\kappa^6} + \frac{(-50)\sin(\kappa(1-u))}{\kappa^{11}}\\
& + \frac{(5 - 50u)\cos(\kappa(1-u))}{\kappa^{10}} + \frac{(-5u + 25u^2)\sin(\kappa(1-u))}{\kappa^9} + \frac{(-\frac{5u^2}{2} + \frac{43u^3}{6})\cos(\kappa(1-u))}{\kappa^8}\\
& + \frac{(\frac{5u^3}{6} - \frac{7u^4}{6})\sin(\kappa(1-u))}{\kappa^7} + \frac{(\frac{u^4}{6} - \frac{u^5}{12})\cos(\kappa(1-u))}{\kappa^6}.
\end{aligned}
$$

Remark 4. If the above ten coefficients are seen as Laurent series in terms of κ, then numerical calculations show that all coefficients for negative κ-powers equal zero. This had to be the case since our expressions are analytic in κ. The latter can be seen from the fact that e.g. the LHS of (25) remains bounded if we let $\kappa \to 0$.

Remark 5. When[2] we put $u = 1$, the limit of A_κ as $\kappa \to 0$ equals $\frac{42}{9!}$. If we let the weight-function $w(t)$ be an approximation to the characteristic function on $[T, 2T]$, then (25) is seen to be consistent with the conjecture that

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^6 \, dt \sim \frac{42}{9!} \cdot a_3 \cdot T \log^9 T \tag{46}$$

which of course is good news[3].

3.3 Obtaining an inequality in terms of κ

It is now time to investigate both sides of Corollary 2. Focusing on the RHS, we are lead to recall (14). Since $w_+(t)$ is supported in $[\frac{T}{2}, 4T]$ we may use that

$$\theta'(t) = \frac{\log T}{2} + O(1). \tag{47}$$

The contribution to the integral in the RHS of (11) coming from the error term in (47) can be seen to be $\ll T \log^{10} T$. To do this we simply use the Cauchy–Schwarz inequality

$$\Big| \int f(x)g(x) \, dx \Big|^2 \leqslant \int |f(x)|^2 \, dx \int |g(x)|^2 \, dx.$$

[2]Although the results in this article (via [9]) only are shown for $u < 1/11$, it may be that they hold for $u < 1$.

[3]However, putting $u = 1/2$ and letting $\kappa \to 0$ does not give half of the sixth power moment.

We can thus via (12) and (14) convert both sides of (11) into expressions involving the integrals in (15)-(24). The latter are of course evaluated using (25)-(34). Explicitly this procedure yields

$$\begin{aligned}
&A_\kappa \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_-(t)\, dt\Big] \cdot \log^9 T + O(T\log^8 T) \qquad (48)\\
&\leqslant \Big(\frac{\kappa}{\pi}\Big)^2 \cdot \Big\{(v/2-u)^2 A_\kappa + (v-2u)B_\kappa + C_\kappa + (v-2u)D_\kappa + (v-2u)E_\kappa\\
&+ F_\kappa + G_\kappa + 2H_\kappa + 2I_\kappa + 2J_\kappa\Big\} \cdot a_3 \cdot \Big[\int_{-\infty}^{\infty} w_+(t)\, dt\Big] \cdot \log^9 T + O(T\log^8 T).
\end{aligned}$$

Immediately from the definitons (9) and (10) one sees

$$w_-(t) \geqslant \exp(-2\eta)\chi_{[T+2T_0,2T-2T_0]}(t) \tag{49}$$

and

$$w_+(t) \leqslant \exp(2\eta)\chi_{[T-T_0,2T+T_0]}(t). \tag{50}$$

Therefore for sufficiently large T we have (recall (8))

$$\int_{-\infty}^{\infty} w_-(t)\, dt \geqslant C_-(\eta)T \tag{51}$$

and

$$\int_{-\infty}^{\infty} w_+(t)\, dt \leqslant C_+(\eta)T, \tag{52}$$

for some (fixed) constants $C_\pm(\eta)$, which can be chosen as close to 1 as we like.

Summarizing, we conclude that IF

$$\begin{aligned}
A_\kappa > \Big(\frac{\kappa}{\pi}\Big)^2 \cdot \Big\{&(v/2-u)^2 A_\kappa + (v-2u)B_\kappa + C_\kappa + (v-2u)D_\kappa + (v-2u)E_\kappa\\
&+ F_\kappa + G_\kappa + 2H_\kappa + 2I_\kappa + 2J_\kappa\Big\}, \qquad (53)
\end{aligned}$$

then we have a contradiction to our Main Assumption. That would imply the existence of a subinterval of $[T, 2T - \frac{\kappa}{\log T}]$ of length at least $\frac{\kappa}{\log T}$, in which the function $t \mapsto P(t, u, v, \kappa)$ has no zeros. A simple proof by contradiction shows that this implies that there must be a subinterval of $[T, 2T]$ of length at least $\frac{2\kappa}{\log T}$, in which the function $t \mapsto \zeta(\frac{1}{2} + it)$ has no zeros.

Using Mathematica, the inequality (53) is seen to hold with $u = 0.0909$, $v = 2.13$ and $\kappa = 8.69$. Thus Theorem 1 holds since $2 \cdot 8.69 > 2.766 \cdot 2\pi$.

Remark 6. If one studies the κ-inequality (53) as $u \to 0$, one sees that (53) is satisfied for $\kappa = 8.264$ (with $v = 2$), yielding gaps of length at least 2.63 times the average. This is effectively what Hall did in [7] (he did not use any amplifier).

Remark 7. For what it is worth, note that if the results (again via [9]) would remain valid for any $u < 1/2$ (whether this is the case or not is unknown), then one could take $u = 0.4999$, $v = 2.68$ and $\kappa = 10.23$ and see that (53) holds. This would imply the existence of gaps of length at least 3.25 times the average. Moreover, $u = 0.55$ and $v = 2.74$ would yield gaps of length at least 3.26 times the average and $u = 0.9999$ and $v = 3$ would yield gaps of length at least 3.05 times the average[4].

Remark 8. As a side-note, it is likely that replacing $M(\frac{1}{2} + it)$ in the definition of our function $P(t, u, v, \kappa)$ in (1) by

$$\sum_{h \leqslant T^u} \frac{A + B\frac{\log(T^u/h)}{\log(T^u)}}{h^{1/2+it}},$$

with some suitable choice of A and B, would have lead to a slightly better gap-result. However, such calculations would be very long.

4 Evaluation of our integrals

4.1 On the article "The twisted fourth moment of the Riemann zeta function"

We will make use of the main theorem in the article "The twisted fourth moment of the Riemann zeta function", written by Hughes and Young [9]. Before we reproduce their result, we must introduce a little bit of notation.

Define

$$A_{\alpha,\beta,\gamma,\delta}(s) = \frac{\zeta(1+s+\alpha+\gamma)\zeta(1+s+\alpha+\delta)\zeta(1+s+\beta+\gamma)\zeta(1+s+\beta+\delta)}{\zeta(2+2s+\alpha+\beta+\gamma+\delta)}. \quad (54)$$

Let

$$\sigma_{\alpha,\beta}(n) = \sum_{n_1 n_2 = n} n_1^{-\alpha} n_2^{-\beta}. \quad (55)$$

[4]Being unable to explain why using $u = 1/2$ leads to bigger gaps than $u = 1$, let me just mention that this was also the case when I (admittedly on rough paper and using ratios conjectures) looked at how amplifying the second moment of the Riemann zeta-function improved Hall's method for finding large gaps.

Next, suppose $(h,k)=1$, $p^{h_p}||h$ and $p^{k_p}||k$, and define

$$B_{\alpha,\beta,\gamma,\delta,h,k}(s)=\prod_{p|h}\left(\frac{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^{j+h_p})p^{-j(s+1)}}{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j(s+1)}}\right)
\times\prod_{p|k}\left(\frac{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^{j+k_p})\sigma_{\gamma,\delta}(p^j)p^{-j(s+1)}}{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j(s+1)}}\right). \tag{56}$$

Then we write

$$Z_{\alpha,\beta,\gamma,\delta,h,k}(s)=A_{\alpha,\beta,\gamma,\delta}(s)B_{\alpha,\beta,\gamma,\delta,h,k}(s). \tag{57}$$

Theorem 3 (Main theorem in [9])**.** *Let*

$$I(h,k)=\int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it}\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta+it)\zeta(\tfrac12+\gamma-it)\zeta(\tfrac12+\delta-it)w(t)\,dt, \tag{58}$$

where $w(t)$ is a smooth, non-negative function with support contained in $[\frac{T}{2},4T]$, satisfying $w^{(j)}(t)\ll_j T_0^{-j}$ for all $j=0,1,2,...,$ where $T^{\frac12+\epsilon}\ll T_0\ll T$. Suppose $(h,k)=1$, $hk\leqslant T^{\frac{2}{11}-\epsilon}$, and that $\alpha,\beta,\gamma,\delta$ are complex numbers $\ll(\log T)^{-1}$. Then

$$\begin{aligned}I(h,k)=\frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty}w(t)\Bigg(&Z_{\alpha,\beta,\gamma,\delta,h,k}(0)+\left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta}Z_{-\gamma,-\delta,-\alpha,-\beta,h,k}(0)\\
&+\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma}Z_{-\gamma,\beta,-\alpha,\delta,h,k}(0)+\left(\frac{t}{2\pi}\right)^{-\alpha-\delta}Z_{-\delta,\beta,\gamma,-\alpha,h,k}(0)\\
&+\left(\frac{t}{2\pi}\right)^{-\beta-\gamma}Z_{\alpha,-\gamma,-\beta,\delta,h,k}(0)+\left(\frac{t}{2\pi}\right)^{-\beta-\delta}Z_{\alpha,-\delta,\gamma,-\beta,h,k}(0)\Bigg)\,dt\\
&+O(T^{\frac34+\epsilon}(hk)^{\frac78}(T/T_0)^{\frac94}).\end{aligned} \tag{59}$$

Brief comment on the proof: The proof is very complicated and the reader is referred to [9]. What follows is just a very brief outline.

The starting point is an approximate functional equation. Let $G(s)$ be an even, entire function of rapid decay as $|s|\to\infty$ in any fixed strip $|\mathrm{Re}(s)|\leqslant C$ and let

$$V_{\alpha,\beta,\gamma,\delta,t}(x)=\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}g_{\alpha,\beta,\gamma,\delta}(s,t)x^{-s}\,ds, \tag{60}$$

where

$$g_{\alpha,\beta,\gamma,\delta}(s,t)=\frac{\Gamma\left(\frac{\frac12+\alpha+s+it}{2}\right)}{\Gamma\left(\frac{\frac12+\alpha+it}{2}\right)}\frac{\Gamma\left(\frac{\frac12+\beta+s+it}{2}\right)}{\Gamma\left(\frac{\frac12+\beta+it}{2}\right)}\frac{\Gamma\left(\frac{\frac12+\gamma+s-it}{2}\right)}{\Gamma\left(\frac{\frac12+\gamma-it}{2}\right)}\frac{\Gamma\left(\frac{\frac12+\delta+s-it}{2}\right)}{\Gamma\left(\frac{\frac12+\delta-it}{2}\right)}. \tag{61}$$

Then

$$\begin{aligned}
&\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta+it)\zeta(\tfrac{1}{2}+\gamma-it)\zeta(\tfrac{1}{2}+\delta-it) \qquad (62)\\
&= \sum_{m,n\geqslant 1} \frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)}{(mn)^{1/2}}\Big(\frac{m}{n}\Big)^{-it} V_{\alpha,\beta,\gamma,\delta,t}(\pi^2 mn)\\
&\qquad + X_{\alpha,\beta,\gamma,\delta,t}\sum_{m,n\geqslant 1} \frac{\sigma_{-\gamma,-\delta}(m)\sigma_{-\alpha,-\beta}(n)}{(mn)^{1/2}}\Big(\frac{m}{n}\Big)^{-it} V_{-\gamma,-\delta,-\alpha,-\beta,t}(\pi^2 mn)\\
&\qquad + O((1+|t|)^{-2007}),
\end{aligned}$$

where

$$X_{\alpha,\beta,\gamma,\delta,t} := \pi^{\alpha+\beta+\gamma+\delta}\frac{\Gamma\Big(\frac{\frac{1}{2}-\alpha-it}{2}\Big)}{\Gamma\Big(\frac{\frac{1}{2}+\alpha+it}{2}\Big)}\frac{\Gamma\Big(\frac{\frac{1}{2}-\beta-it}{2}\Big)}{\Gamma\Big(\frac{\frac{1}{2}+\beta+it}{2}\Big)}\frac{\Gamma\Big(\frac{\frac{1}{2}-\gamma+it}{2}\Big)}{\Gamma\Big(\frac{\frac{1}{2}+\gamma-it}{2}\Big)}\frac{\Gamma\Big(\frac{\frac{1}{2}-\delta+it}{2}\Big)}{\Gamma\Big(\frac{\frac{1}{2}+\delta-it}{2}\Big)}. \qquad (63)$$

Using the approximate functional equation (62) in the definition of $I(h,k)$ (see (58)) yields

$$\begin{aligned}
I(h,k) &= \sum_{m,n\geqslant 1} \frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)}{(mn)^{1/2}}\int_{-\infty}^{\infty}\Big(\frac{hm}{kn}\Big)^{-it} V_{\alpha,\beta,\gamma,\delta,t}(\pi^2 mn)w(t)\,dt \qquad (64)\\
&\quad + \sum_{m,n\geqslant 1}\Big\{\frac{\sigma_{-\gamma,-\delta}(m)\sigma_{-\alpha,-\beta}(n)}{(mn)^{1/2}}\\
&\quad \times \int_{-\infty}^{\infty}\Big(\frac{hm}{kn}\Big)^{-it} X_{\alpha,\beta,\gamma,\delta,t}V_{-\gamma,-\delta,-\alpha,-\beta,t}(\pi^2 mn)w(t)\,dt\Big\} + O(1).
\end{aligned}$$

The two main terms in (64) can be treated similarly. Denoting the first one by $I^{(1)}(h,k)$, then upon opening up the integral formula for V, one finds that

$$\begin{aligned}
I^{(1)}(h,k) &= \sum_{m,n\geqslant 1}\Big\{\frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)}{(mn)^{1/2}} \qquad (65)\\
&\times \frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}(\pi^2 mn)^{-s}\int_{-\infty}^{\infty}\Big(\frac{hm}{kn}\Big)^{-it} g_{\alpha,\beta,\gamma,\delta}(s,t)w(t)\,dt\,ds\Big\}.
\end{aligned}$$

The authors of [9] split the sum in (65) into the diagonal part corresponding to the terms for which $hm = kn$ and the non-diagonal part (the other terms). Whereas it is relatively simple to treat the diagonal contribution, it was a nice achievement to be able to treat the more complicated off-diagonal terms (results from the article [3] by Duke, Friedlander and Iwaniec are used and the latter part of the proof of Theorem 3 involves a lot of simplifying).

Remark 9. Throughout this article we will always use the choice $T_0 = T^{1-\epsilon}$ and the practice of letting ϵ stand for a small positive number, not necessarily always the same.

Remark 10. It is not immediately obvious that the main term in (59) is an analytic function in terms of the shifts (e.g. $Z_{\alpha,\beta,\gamma,\delta}(0)$ has singularities at $\alpha = -\gamma$, $\alpha = -\delta$, $\beta = -\gamma$, $\beta = -\delta$), however, due to nice cancellation, analyticity holds. Lemma 2.5.1 in the article [1] by Conrey, Farmer, Keating, Rubinstein and Snaith is very helpful when showing this. In Section 4.4 we will carry out the details in a similar situation.

4.2 Initial step in using Theorem 3

Suppose that $w(t)$ is a function satisfying the assumptions in Theorem 3, given the choice $T_0 = T^{1-\epsilon}$. With

$$M_1(s) = \sum_{h \leqslant T^u} \frac{a_1(h)}{h^s} \tag{66}$$

and

$$M_2(s) = \sum_{k \leqslant T^u} \frac{a_2(k)}{k^s}, \tag{67}$$

where $0 < u < 1/11$ and the a_i-coefficients are real, we will want to asymptotically evaluate expressions such as

$$\int_{-\infty}^{\infty} M_1(\tfrac{1}{2}+it)M_2(\tfrac{1}{2}-it)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta+it)\zeta(\tfrac{1}{2}+\gamma-it)\zeta(\tfrac{1}{2}+\delta-it)w(t)\,dt. \tag{68}$$

By expanding out M_1 and M_2 we obtain

$$\sum_{h,k \leqslant T^u} \frac{a_1(h)a_2(k)}{\sqrt{hk}} \int_{-\infty}^{\infty} \left(\frac{h}{k}\right)^{-it} \zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta+it)\zeta(\tfrac{1}{2}+\gamma-it)\zeta(\tfrac{1}{2}+\delta-it)w(t)\,dt. \tag{69}$$

Let us write (59) as

$$I(h,k) = J(h,k) + E(h,k). \tag{70}$$

Then (69) equals

$$\begin{aligned}
&\sum_{h,k \leqslant T^u} \frac{a_1(h)a_2(k)}{\sqrt{hk}} I(h,k) \\
&= \sum_{m \leqslant T^u} \frac{1}{m} \sum_{\substack{h,k \leqslant T^u/m \\ (h,k)=1}} \frac{a_1(hm)a_2(km)}{\sqrt{hk}} I(h,k) \\
&= \sum_{m \leqslant T^u} \frac{1}{m} \sum_{\substack{h,k \leqslant T^u/m \\ (h,k)=1}} \frac{a_1(hm)a_2(km)}{\sqrt{hk}} J(h,k) + \sum_{m \leqslant T^u} \frac{1}{m} \sum_{\substack{h,k \leqslant T^u/m \\ (h,k)=1}} \frac{a_1(hm)a_2(km)}{\sqrt{hk}} E(h,k).
\end{aligned} \tag{71}$$

The first term in (71) clearly equals

$$\begin{aligned}
&\sum_{m\leqslant T^u}\frac{1}{m}\sum_{h,k\leqslant T^u/m}\frac{a_1(hm)a_2(km)}{\sqrt{hk}}J(h,k)\sum_{d|(h,k)}\mu(d)\\
&=\sum_{m\leqslant T^u}\frac{1}{m}\sum_{d\leqslant T^u/m}\frac{\mu(d)}{d}\sum_{h,k\leqslant T^u/md}\frac{a_1(hmd)a_2(kmd)}{\sqrt{hk}}J(hd,kd)\\
&=\sum_{d\leqslant T^u}\frac{\mu(d)}{d}\sum_{m\leqslant T^u/d}\frac{1}{m}\sum_{h,k\leqslant T^u/md}\frac{a_1(hmd)a_2(kmd)}{\sqrt{hk}}J(hd,kd). \qquad (72)
\end{aligned}$$

The second term in (71) is

$$\ll T^{\frac{3}{4}+\epsilon}(T/T_0)^{\frac{9}{4}}\sum_{m\leqslant T^u}\Big\{\frac{1}{m}\sum_{h\leqslant T^u/m}|a_1(hm)|h^{3/8}\sum_{k\leqslant T^u/m}|a_2(km)|k^{3/8}\Big\}. \qquad (73)$$

4.3 Specialising on the most standard case

Let us now investigate the expression in (68) when $a_1(h)=a_2(k)=1$ for all values of h and k. Our goal is to simplify the main term (which will turn out to be of order $T\log^9 T$) as much as possible, treating anything which is $\ll T\log^8 T$ as an error term.

As noticed in Section 4.2, (68) splits up into a main term (72) and an error term (73). The latter is

$$\ll T^{\frac{3}{4}+\epsilon}(T/T_0)^{\frac{9}{4}}\sum_{m\leqslant T^u}\Big\{\frac{1}{m}\cdot(T^u/m)^{11/8}\cdot(T^u/m)^{11/8}\Big\}\ll T^{\frac{3}{4}+\epsilon}(T)^{\frac{9\epsilon}{4}}T^{\frac{11u}{4}}\ll T,$$

recalling Remark 9 for the second step and the last step being true upon choosing ϵ to be sufficiently small. The main term (72) is

$$\begin{aligned}
&\sum_{d\leqslant T^u}\frac{\mu(d)}{d}\sum_{m\leqslant T^u/d}\frac{1}{m}\sum_{h,k\leqslant T^u/md}\frac{J(hd,kd)}{\sqrt{hk}}\\
&=\sum_{d\leqslant T^u}\frac{\mu(d)}{d^2}\sum_{m\leqslant T^u/d}\frac{1}{m}\int_{-\infty}^{\infty}w(t)\sum_{h,k\leqslant T^u/md}\frac{1}{hk}\Big\{...\Big\}\,dt, \qquad (74)
\end{aligned}$$

where the expression $\{...\}$ in (74) stands for

$$\begin{aligned}
&Z_{\alpha,\beta,\gamma,\delta,hd,kd}(0)+\Big(\frac{t}{2\pi}\Big)^{-\alpha-\beta-\gamma-\delta}Z_{-\gamma,-\delta,-\alpha,-\beta,hd,kd}(0)\\
&+\Big(\frac{t}{2\pi}\Big)^{-\alpha-\gamma}Z_{-\gamma,\beta,-\alpha,\delta,hd,kd}(0)+\Big(\frac{t}{2\pi}\Big)^{-\alpha-\delta}Z_{-\delta,\beta,\gamma,-\alpha,hd,kd}(0)\\
&+\Big(\frac{t}{2\pi}\Big)^{-\beta-\gamma}Z_{\alpha,-\gamma,-\beta,\delta,hd,kd}(0)+\Big(\frac{t}{2\pi}\Big)^{-\beta-\delta}Z_{\alpha,-\delta,\gamma,-\beta,hd,kd}(0). \qquad (75)
\end{aligned}$$

4.4 Studying $\mathcal{Q}_{A,B}(T_1, T_2, f_1, f_2)$

As we shall see in the Sections 4.5 and 4.6, it is possible to simplify our main term (74) considerably. However, we first need to introduce and become familiar with a bit of new notation.

Suppose that $f_1(x_1, x_2, T_2)$ and $f_2(x_1, x_2, T_2)$ are functions which are analytic and symmetric in their complex variables x_1 and x_2. Also, let A and B be sets of complex numbers with $|A| = |B| = 2$ and write them as

$$A := \{\alpha_1, \alpha_2\} \tag{76}$$

and

$$B := \{\alpha_3, \alpha_4\}. \tag{77}$$

We then define

$$\mathcal{Q}_{A,B}(T_1, T_2, f_1, f_2) := \sum_{\substack{R\subseteq A \\ S\subseteq B \\ |R|=|S|}} \mathcal{Q}((A \setminus R) \cup (-S), (B \setminus S) \cup (-R), T_1, T_2, f_1, f_2), \tag{78}$$

where we by $-U$ mean $\{-u : u \in U\}$ and

$$\mathcal{Q}(X, Y, T_1, T_2, f_1, f_2) := T_1^{(\delta_X+\delta_Y)/2} F_1(X, T_2) F_2(Y, T_2) \prod_{\substack{x\in X \\ y\in Y}} \frac{1}{(x + y)}, \tag{79}$$

with

$$\delta_X := \sum_{x\in X} x \tag{80}$$

and where for $X = \{x_1, x_2\}$ we let

$$F_i(X, T_2) := f_i(x_1, x_2, T_2),\ i = 1, 2. \tag{81}$$

Next let Ξ denote the set of $\binom{4}{2}$ permutations $\sigma \in S_4$ satisfying $\sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(4)$. Let us for $\sigma \in \Xi$ define

$$\begin{aligned} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)}) &= K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)}, T_1, T_2, f_1, f_2) \\ &:= \mathcal{Q}(\{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}\}, \{-\alpha_{\sigma(3)}, -\alpha_{\sigma(4)}\}, T_1, T_2, f_1, f_2). \end{aligned}$$

Then one has that

$$\sum_{\sigma\in\Xi} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)}) = \mathcal{Q}_{A,-B}(T_1, T_2, f_1, f_2). \tag{82}$$

We now show that the LHS of (82) is an analytic function of the α-shifts. The problem is where any of the relevant $K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)})$ has a singularity. However, we will now show that these singularities must be removable. Suppose that

$$\alpha_i \neq \alpha_j, \text{for } i \neq j. \tag{83}$$

From Lemma 2.5.1 in [1] we then have the following formula:

$$\begin{aligned}&\sum_{\sigma\in\Xi} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)}) \\ &= \frac{1}{(2!)^2}\frac{1}{(2\pi i)^4}\oint\cdots\oint \frac{K(z_1,z_2;z_3,z_4)\Delta(z_1,...,z_4)^2}{\prod_{i=1}^4\prod_{j=1}^4(z_i-\alpha_j)}\,dz_1...dz_4,\end{aligned} \tag{84}$$

where

$$\Delta(z_1,...,z_4) := \prod_{1\leqslant i<j\leqslant 4}(z_j - z_i),$$

and where one integrates about circles enclosing the α_j's. By choosing the radii of the circles to be suitably large[5], we obtain an upper bound for the RHS of (84). The function $\sum_{\sigma\in\Xi} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)})$ thus remains bounded whenever (83) is satisfied. This allows us to conclude[6] that the possible singularities must be removable. Hence $\mathcal{Q}_{A,-B}(T_1,T_2,f_1,f_2)$ is analytic by (82), which in turn implies that $\mathcal{Q}_{A,B}(T_1,T_2,f_1,f_2)$ is an analytic function of the shifts.

4.5 Initial simplification of Theorem 3 in the most standard case

Theorem 4. *Suppose that $w(t)$ is a smooth, non-negative function with support contained in $[\frac{T}{2}, 4T]$ satisfying $w^{(j)}(t) \ll_j (T^{1-\epsilon})^{-j}$ for all $j = 0,1,2,...$, that $\alpha,\beta,\gamma,\delta$ are complex numbers $\ll (\log T)^{-1}$ and let*

$$M(s) = \sum_{h\leqslant T^u}\frac{1}{h^s},$$

with $0 < u < 1/11$. Then

$$\begin{aligned}&\int_{-\infty}^{\infty}|M(\tfrac12+it)|^2\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta+it)\zeta(\tfrac12+\gamma-it)\zeta(\tfrac12+\delta-it)w(t)\,dt \\ &= \Big[\int_{-\infty}^{\infty}w(t)\,dt\Big]\cdot a_3\cdot T^{-(\alpha+\beta+\gamma+\delta)/2}\cdot\sum_{m\leqslant T^u}\frac{1}{m}\mathcal{Q}_{A,B}(T,T^u/m,f,f) + O(T\log^8 T),\end{aligned} \tag{85}$$

[5] If analyticity is to be shown for say $|\alpha_i| \leqslant C$, then we can pick the radii of the circles to be $3C$ since then $|z_i - \alpha_j|^{-1} \leqslant 1/C$.

[6] By Riemann's Extension Theorem — see for example [5].

with

$$a_3 = \prod_p \left\{ \left(1 + \frac{4}{p} + \frac{1}{p^2}\right)\left(1 - \frac{1}{p}\right)^4 \right\}$$

and where we are here taking (see also definition given in (79))

$$A := \{\alpha, \beta\}, \tag{86}$$

$$B := \{\gamma, \delta\} \tag{87}$$

and[7]

$$f(x_1, x_2, T_2) := \frac{1}{x_1 x_2} - \frac{T_2^{-x_1}}{x_1(x_2 - x_1)} - \frac{T_2^{-x_2}}{x_2(x_1 - x_2)}. \tag{88}$$

Remark 11. It is easy to show that $f(x_1, x_2, T_2)$ is analytic in x_1 and x_2. Assume (without loss of generality) that $|x_1|, |x_2| \leqslant C$ say. Consider

$$\frac{1}{2\pi i} \int_R \frac{T_2^s}{s(s + x_1)(s + x_2)}\, ds, \tag{89}$$

with R denoting a counter-clockwise integral-contour around a square with vertices at $\pm 2C \pm 2Ci$. Suppose that x_1 and x_2 are different and non-zero. Then (89) equals (88) by Cauchy's Residue Theorem, and the integral in (89) is obviously bounded. It follows that any possible singularities of $f(x_1, x_2, T_2)$ must be removable.

Remark 12. By Section 4.4 we thus know that $\mathcal{Q}_{A,B}(T_1, T_2, f, f)$ is analytic in terms of the shifts. Hence the main term in the RHS of (85) is analytic in the shifts.

Proof. We will first proceed under the assumption that for some fixed constant $C > 0$ we have that

$$|\alpha_i| \geqslant C/\log T, \quad |\alpha_i + \alpha_j| \geqslant C/\log T \quad \text{and} \quad |\alpha_i - \alpha_j| \geqslant C/\log T, \tag{90}$$

where α_i and α_j (with i and j distinct) stand for any of the shifts.

Let us begin by noticing that there is an obvious identification to be made between the terms in (74) and in the main term of (85) (indeed both expressions involve the same number of terms, namely six)[8]. We shall prove Theorem 4 under the assumption (90) by treating each term in (74) individually. Below we will focus on the third term in (74), which will correspond (recalling (78)) to the case $R = \{\alpha\}$ and $S = \{\gamma\}$ in (85). The other terms can be treated analogously.

[7] When comparing (86) with (76), the reader will probably find it easiest to just identify α and β with α_1 and α_2 respectively. Note however that due to symmetry, it does not matter if the roles are reversed.

[8] The reader may find it helpful to factor out $(\frac{t}{2\pi})^{-(\alpha+\beta+\gamma+\delta)/2}$ in (75).

Our starting point will thus be given by

$$\Big\{\int_{-\infty}^{\infty} w(t)\Big(\frac{t}{2\pi}\Big)^{-\alpha-\gamma}\,dt\Big\}\sum_{d\leqslant T^u}\frac{\mu(d)}{d^2}\sum_{m\leqslant T^u/d}\frac{1}{m}\sum_{h,k\leqslant T^u/md}\frac{Z_{-\gamma,\beta,-\alpha,\delta,hd,kd}(0)}{hk}. \tag{91}$$

Let us write (recall (57))

$$Z_{\alpha,\beta,\gamma,\delta,h,k}(0) = A_{\alpha,\beta,\gamma,\delta}(0)B_{\alpha,\beta,\gamma,\delta}(h)E_{\alpha,\beta,\gamma,\delta}(k), \tag{92}$$

with

$$B_{\alpha,\beta,\gamma,\delta}(h) := \prod_{p|h}\left(\frac{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^{j+h_p})p^{-j}}{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j}}\right) \tag{93}$$

and

$$E_{\alpha,\beta,\gamma,\delta}(k) := \prod_{p|k}\left(\frac{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^{j+k_p})\sigma_{\gamma,\delta}(p^j)p^{-j}}{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j}}\right). \tag{94}$$

Using this notation, (91) becomes

$$\begin{aligned}&\Big\{\int_{-\infty}^{\infty} w(t)\Big(\frac{t}{2\pi}\Big)^{-\alpha-\gamma}\,dt\Big\}A_{-\gamma,\beta,-\alpha,\delta}(0)\\ &\times\sum_{d\leqslant T^u}\frac{\mu(d)}{d^2}\sum_{m\leqslant T^u/d}\Big\{\frac{1}{m}\sum_{h\leqslant T^u/md}\frac{B_{-\gamma,\beta,-\alpha,\delta}(hd)}{h}\sum_{k\leqslant T^u/md}\frac{E_{-\gamma,\beta,-\alpha,\delta}(kd)}{k}\Big\}.\end{aligned} \tag{95}$$

It would be desirable to have more information about the two innermost sums in (95). They are similar. Let us study $\sum_{h\leqslant T^u/md}\frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h}$. First note that

$$B_{\alpha,\beta,\gamma,\delta}(w) = \prod_{p|w}B_{\alpha,\beta,\gamma,\delta}(p^{w_p}) \ll \prod_{p|w}w_p p^{\epsilon w_p} \ll \prod_{p|w}p^{\epsilon w_p}\cdot p^{\epsilon w_p} = w^{2\epsilon}. \tag{96}$$

Now define for $\mathrm{Re}(s) > 1$,

$$F(s,\alpha,\beta,\gamma,\delta,d) := \sum_{h=1}^{\infty}\frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h^s}. \tag{97}$$

Suppose for the time being that $\frac{T^u}{md}$ is a half-integer, say $\frac{T^u}{md} = M + \frac{1}{2}$ for some positive integer M. We claim that then

$$\sum_{h\leqslant T^u/md}\frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h} = \frac{1}{2\pi i}\int_{\epsilon-iW}^{\epsilon+iW}\frac{F(1+s,\alpha,\beta,\gamma,\delta,d)(T^u/md)^s}{s}\,ds + O(d^{\epsilon}), \tag{98}$$

with (somewhat arbitrary choice)

$$W = 10\times(T^u/md)^{1.1}. \tag{99}$$

It is easy to show (this is one version of Perron's formula) that

$$\frac{1}{2\pi i}\int_{\epsilon-iW}^{\epsilon+iW}\frac{x^s}{s}\,ds = H(x) + O\Big(\frac{x^\epsilon}{W|\log x|}\Big), \tag{100}$$

for $x > 0$, $x \neq 1$, where

$$H(x) = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } x < 1. \end{cases} \tag{101}$$

Expanding out $F(1+s,\alpha,\beta,\gamma,\delta,d)$ in (98), we thus get

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\epsilon-iW}^{\epsilon+iW}\frac{F(1+s,\alpha,\beta,\gamma,\delta,d)(T^u/md)^s}{s}\,ds \\ &= \sum_{h\leqslant T^u/md}\frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h} + O\Big(\sum_{h=1}^{\infty}\frac{|B_{\alpha,\beta,\gamma,\delta}(hd)|}{h}\cdot\frac{(T^u/mdh)^\epsilon}{W|\log(T^u/mdh)|}\Big). \end{aligned} \tag{102}$$

If $h \notin [\frac{T^u}{2md}, \frac{3T^u}{2md}]$, then $|\log(T^u/mdh)|^{-1} \ll 1$. The part of the error in (102) corresponding to such values of h is thus

$$\ll \sum_{h=1}^{\infty}\frac{(hd)^{\epsilon/2}}{h}\cdot\frac{(T^u/mdh)^\epsilon}{W} \ll d^\epsilon.$$

For $M + \frac{1}{2} = \frac{T^u}{md} < h \leqslant \frac{3T^u}{2md}$, we write

$$h = M + R, \qquad R = 1, ..., \Big[\frac{3T^u}{2md}\Big] - M$$

and spot that here

$$|\log(T^u/mdh)|^{-1} \ll \frac{M}{R},$$

so that the part of the error in (102) corresponding to these values of h is certainly

$$\ll \frac{d^\epsilon}{W}\sum_{R=1}^{M}\frac{M}{R} \ll \frac{d^\epsilon}{W}\cdot M\log(M+1) \ll d^\epsilon.$$

A very similar argument applies when $\frac{T^u}{2md} \leqslant h \leqslant \frac{T^u}{md}$, which concludes the proof of the claim in the case when $\frac{T^u}{md}$ is a half-integer. If this is not the case, then certainly for some $0 < \mu < 1$ we have that $\frac{T^u}{md} + \mu$ is a half-integer. We obtain[9]

$$\begin{aligned} \sum_{h\leqslant T^u/md}\frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h} &= \sum_{h\leqslant T^u/md+\mu}\frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h} + O(d^\epsilon) \\ &= \frac{1}{2\pi i}\int_{\epsilon-iW}^{\epsilon+iW}\frac{F(1+s,\alpha,\beta,\gamma,\delta,d)(T^u/md+\mu)^s}{s}\,ds + O(d^\epsilon). \end{aligned} \tag{103}$$

[9]By looking at the derivation of (98), it is obvious that introducing μ does not necessitate a change in our choice of W.

We are thus lead to investigating the RHS of (103). By contemplating the definition of $B_{\alpha,\beta,\gamma,\delta}(hd)$, one concludes (see (106)-(108)) that one may write

$$F(s,\alpha,\beta,\gamma,\delta,d) =: \zeta(s+\gamma)\zeta(s+\delta)G(s,\alpha,\beta,\gamma,\delta,d), \tag{104}$$

with $G(s,\alpha,\beta,\gamma,\delta,d)$ being an analytic function for $\mathrm{Re}(s) > 1/2$. Let us in the following discussion restrict ourselves to when $0.9 \leqslant \mathrm{Re}(s) \leqslant 1.1$. We have

$$G(s,\alpha,\beta,\gamma,\delta,d) = \prod_{p|d}\left\{(1-p^{-s-\gamma})(1-p^{-s-\delta})\right\} \sum_{\substack{h=1\\ p|h\Rightarrow p|d}}^{\infty} \frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h^s}$$
$$\times \prod_{p\nmid d}\left\{(1-p^{-s-\gamma})(1-p^{-s-\delta}) \sum_{M=0}^{\infty} \frac{B_{\alpha,\beta,\gamma,\delta}(p^M)}{p^{Ms}}\right\}.$$

Let us write this as

$$G(s,\alpha,\beta,\gamma,\delta,d) = G_1(s,\alpha,\beta,\gamma,\delta,d) \times G_2(s,\alpha,\beta,\gamma,\delta,d). \tag{105}$$

We notice that (uniformly)

$$B_{\alpha,\beta,\gamma,\delta}(1) = 1, \tag{106}$$

$$B_{\alpha,\beta,\gamma,\delta}(p) = p^{-\gamma} + p^{-\delta} + O(p^{-0.9+2\epsilon}) \tag{107}$$

and

$$B_{\alpha,\beta,\gamma,\delta}(p^N) = O((N+1)p^{N\epsilon}), N \geqslant 2. \tag{108}$$

Thus

$$\sum_{M=0}^{\infty} \frac{B_{\alpha,\beta,\gamma,\delta}(p^M)}{p^{Ms}} = 1 + p^{-s-\gamma} + p^{-s-\delta} + O(p^{-1.8+2\epsilon})$$

and hence

$$G_2(s,\alpha,\beta,\gamma,\delta,d) = \prod_{p\nmid d}(1+O(p^{-1.8+2\epsilon})).$$

In particular we obtain

$$G_2(s,\alpha,\beta,\gamma,\delta,d) \ll 1.$$

The main thing to notice in the above argument is that the terms of order (roughly) $p^{-\mathrm{Re}(s)}$ exactly cancel in the factors of $G_2(s,\alpha,\beta,\gamma,\delta,d)$. Keeping this in mind, one can show

$$\frac{\partial G_2(s,\alpha,\beta,\gamma,\delta,d)}{\partial x} \ll \sum_{p\nmid d} \log p \cdot p^{-1.8+2\epsilon} \cdot 1 \ll 1,$$

where x here shall mean any of $\alpha,\beta,\gamma,\delta,s$.

Now we study $G_1(s,\alpha,\beta,\gamma,\delta,d)$. First of all,

$$G_{11}(s,\alpha,\beta,\gamma,\delta,d) := \prod_{p|d}\{(1-p^{-s-\gamma})(1-p^{-s-\delta})\} \ll \prod_{p|d}\{2\cdot 2\} \ll d^{\epsilon}.$$

Also, logarithmic differentiation yields

$$\frac{\partial G_{11}(s,\alpha,\beta,\gamma,\delta,d)}{\partial x} \ll d^{\epsilon}.$$

Next, we study

$$G_{12}(s,\alpha,\beta,\gamma,\delta,d) := \sum_{\substack{h=1\\ p|h\Rightarrow p|d}}^{\infty} \frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h^s}.$$

Recalling (96) we find

$$\begin{aligned} G_{12}(s,\alpha,\beta,\gamma,\delta,d) \ll d^{\epsilon} \sum_{\substack{h=1\\ p|h\Rightarrow p|d}}^{\infty} h^{-0.9+\epsilon} &\ll d^{\epsilon}\prod_{p|d}\sum_{j=0}^{\infty} p^{j(-0.9+\epsilon)} \\ &\ll d^{\epsilon}\prod_{p|d}(1+O(p^{-0.9+\epsilon})) \ll d^{\epsilon}. \end{aligned}$$

Also, although the details are somewhat more delicate, one can in a straight-forward direct way do a similar upper bound calculation as above in order to deduce that

$$\frac{\partial G_{12}(s,\alpha,\beta,\gamma,\delta,d)}{\partial x} \ll d^{\epsilon}.$$

Putting things together we have

$$G(s,\alpha,\beta,\gamma,\delta,d) \ll d^{\epsilon} \tag{109}$$

and

$$\frac{\partial G(s,\alpha,\beta,\gamma,\delta,d)}{\partial x} \ll d^{\epsilon}. \tag{110}$$

Now it is time to go back to (103) which tells us that

$$\begin{aligned} &\sum_{h\leqslant T^u/md} \frac{B_{-\gamma,\beta,-\alpha,\delta}(hd)}{h} \\ &= \frac{1}{2\pi i}\int_{\epsilon-iW}^{\epsilon+iW} \frac{\zeta(1-\alpha+s)\zeta(1+\delta+s)(T^u/md+\mu)^s G(1+s,-\gamma,\beta,-\alpha,\delta,d)}{s}\, ds + O(d^{\epsilon}). \end{aligned} \tag{111}$$

Using a rectangular path, we move the line of integration from $\mathrm{Re}(s)=\epsilon$ to $\mathrm{Re}(s)=-0.05$. The contribution along any of the two horizontal line-segments is

$$\ll \int_{-0.05}^{\epsilon} \frac{(W^{0.1\times\frac{1}{6}+\epsilon})^2 (T^u/md+\mu)^{\epsilon} d^{\epsilon}}{W}\, dx \ll d^{\epsilon}.$$

And the contribution along the new vertical line-segment is

$$\begin{aligned}\int_{-0.05-iW}^{-0.05+iW} &= \int_{-0.05-iW}^{-0.05-2i} + \int_{-0.05-2i}^{-0.05+2i} + \int_{-0.05+2i}^{-0.05+iW} \\ &\ll 2\int_2^W \frac{(t^{0.1\times\frac{1}{6}+\epsilon})^2 (T^u/md+\mu)^{-0.05} d^\epsilon}{t}\, dt + \int_{-2}^{2} \frac{(T^u/md+\mu)^{-0.05} d^\epsilon}{0.05}\, dt \\ &\ll (T^u/md)^{-0.05} d^\epsilon \Big\{ \int_2^W t^{-\frac{29}{30}+\epsilon}\, dt + 1 \Big\} \\ &\ll d^\epsilon.\end{aligned}$$

By Cauchy's Residue Theorem

$$\sum_{h\leqslant T^u/md} \frac{B_{-\gamma,\beta,-\alpha,\delta}(hd)}{h} = \mathrm{Res}_{s=0} + \mathrm{Res}_{s=\alpha} + \mathrm{Res}_{s=-\delta} + O(d^\epsilon), \tag{112}$$

where of course we are referring to the residues of the integrand in (111).

Finally we have (almost) got all the puzzle-pieces needed to conclude the proof under the assumption (90). We consider the various factors in (95). First of all

$$\begin{aligned}\int_{-\infty}^{\infty} w(t)\Big(\frac{t}{2\pi}\Big)^{-\alpha-\gamma} dt &= T^{-\alpha-\gamma}\int_{T/2}^{4T} w(t)\exp\Big\{(-\alpha-\gamma)\log\Big(\frac{t}{2\pi T}\Big)\Big\}\, dt \\ &= T^{-\alpha-\gamma}\int_{T/2}^{4T} w(t)\Big\{1+O\Big(\frac{1}{\log T}\Big)\Big\}\, dt \\ &= T^{-\alpha-\gamma}\int_{-\infty}^{\infty} w(t)\, dt + O\Big(1\cdot\int_{T/2}^{4T}\frac{|w(t)|}{\log T}\, dt\Big) \\ &= T^{-\alpha-\gamma}\int_{-\infty}^{\infty} w(t)\, dt + O\Big(\frac{T}{\log T}\Big) \\ &= T^{-(\alpha+\beta+\gamma+\delta)/2}\cdot T^{(\beta-\gamma+\delta-\alpha)/2}\cdot\int_{-\infty}^{\infty} w(t)\, dt + O\Big(\frac{T}{\log T}\Big), \end{aligned} \tag{113}$$

where we for future need also remark that the main term in (113) trivially is $\ll T$.

Secondly,

$$\begin{aligned} A_{-\gamma,\beta,-\alpha,\delta}(0) &= \frac{\zeta(1-\gamma-\alpha)\zeta(1-\gamma+\delta)\zeta(1+\beta-\alpha)\zeta(1+\beta+\delta)}{\zeta(2-\gamma+\beta-\alpha+\delta)} \\ &= \frac{1}{\zeta(2)}\cdot\frac{1}{(-\gamma-\alpha)}\cdot\frac{1}{(-\gamma+\delta)}\cdot\frac{1}{(\beta-\alpha)}\cdot\frac{1}{(\beta+\delta)} + O(\log^3 T), \end{aligned} \tag{114}$$

where we remark that the main term in (114) obviously is of order $\log^4 T$.

Thirdly, the m-summands in (95) contain two sums like in (112). Each of those sums will be evaluated by using (112) and hence give rise to residues. We will now explain how to treat the residue at $s=\alpha$ in (112). Explicitly the residue is

$$\frac{\zeta(1+\delta+\alpha)(T^u/md+\mu)^\alpha G(1+\alpha,-\gamma,\beta,-\alpha,\delta,d)}{\alpha}. \tag{115}$$

Repeatedly using the Theorem of Calculus

$$F(\gamma(b)) - F(\gamma(a)) = \int_{\gamma} F'(z)\, dz,$$

we obtain, recalling (110), that

$$G(1+\alpha, -\gamma, \beta, -\alpha, \delta, d) = G(1,0,0,0,0,d) + O\Big(\frac{d^{\epsilon}}{\log T}\Big). \tag{116}$$

We easily deduce that (115) is

$$\frac{(T^u/md)^{\alpha} G(1,0,0,0,0,d)}{(\delta+\alpha)\alpha} + O(d^{\epsilon}\log T), \tag{117}$$

where clearly the main term in (117) is

$$\ll d^{\epsilon}\log^2 T.$$

The other five residues are treated similarly. Then we put in our new expressions for $\sum_{h\leqslant T^u/md} \frac{B_{-\gamma,\beta,-\alpha,\delta}(hd)}{h}$ and $\sum_{k\leqslant T^u/md} \frac{E_{-\gamma,\beta,-\alpha,\delta}(kd)}{k}$ into (95). Since the outer sum over d in (95) always will be convergent (due to the presence of d^2 in the denominator), an inspection yields two things. First of all that if we ever choose to take the error-part of any of the discussed expressions, we will in (95) end up with something that is $\ll T\log^8 T$ and which thus can be relegated to the error term in (85). And secondly that we may replace the $(T^u/md)^{\alpha}$ in (117) by $(T^u/m)^{\alpha}$ and change the range of summation over m in (95) from $m \leqslant T^u/d$ to $m \leqslant T^u$, since the change introduced by these is $\ll T\log^8 T$.

All-in-all we get that (95) equals

$$\begin{aligned}&\Big[\int_{-\infty}^{\infty} w(t)\, dt\Big] \cdot \frac{1}{\zeta(2)} \cdot \sum_{d\leqslant T^u} \frac{\mu(d)G(1,0,0,0,0,d)^2}{d^2} \\ &\cdot T^{-(\alpha+\beta+\gamma+\delta)/2} \cdot \sum_{m\leqslant T^u} \frac{1}{m}\mathcal{Q}(\{\beta,-\gamma\},\{\delta,-\alpha\},T,T^u/m,f,f) + O(T\log^8 T).\end{aligned} \tag{118}$$

We may extend the finite sum over d in (118) to an infinite one, since

$$\sum_{d>T^u} \frac{\mu(d)G(1,0,0,0,0,d)^2}{d^2} \ll \sum_{d>T^u} \frac{1\cdot d^{2\epsilon}}{d^2} \ll \frac{1}{\log T}. \tag{119}$$

It then remains to establish the identity

$$\frac{1}{\zeta(2)}\sum_{d=1}^{\infty} \frac{\mu(d)G(1,0,0,0,0,d)^2}{d^2} = \prod_p \Big\{\Big(1+\frac{4}{p}+\frac{1}{p^2}\Big)\Big(1-\frac{1}{p}\Big)^4\Big\}. \tag{120}$$

By an explicit calculation

$$G(1,0,0,0,0,1) = \prod_p \Big\{(1-p^{-1})^2 \sum_{M=0}^{\infty} \frac{B_{0,0,0,0}(p^M)}{p^M}\Big\} = \prod_p \Big\{\frac{(1-p^{-1})(1+2p^{-1})}{(1+p^{-1})}\Big\}. \quad (121)$$

It is natural to define

$$g(1,0,0,0,0,d) := G(1,0,0,0,0,d)/G(1,0,0,0,0,1) \quad (122)$$

and next we show that $g(1,0,0,0,0,d)$ is a multiplicative function. Let D_1, $D_2 \in \mathbb{N}$ be relatively prime. Recall (105) and note that $B_{0,0,0,0}(n)$ is a multiplicative function. By expanding out the terms involved in the relation

$$g(1,0,0,0,0,D_1) \cdot g(1,0,0,0,0,D_2) = g(1,0,0,0,0,D_1 D_2),$$

this equality is, after cancellation, seen to be equivalent to

$$\sum_{\substack{h=1\\ p|h \Rightarrow p|D_1}}^{\infty} \frac{B_{0,0,0,0}(hD_1)}{h} \cdot \sum_{\substack{k=1\\ p|k \Rightarrow p|D_2}}^{\infty} \frac{B_{0,0,0,0}(kD_2)}{k} = \sum_{\substack{m=1\\ p|m \Rightarrow p|D_1 D_2}}^{\infty} \frac{B_{0,0,0,0}(mD_1 D_2)}{m}.$$

This identity can be seen to hold by equalling denominators (using multiplicativity of the function $B_{0,0,0,0}(n)$).

Therefore the LHS of (120) equals

$$\frac{G(1,0,0,0,0,1)^2}{\zeta(2)} \sum_{d=1}^{\infty} \frac{\mu(d) g(1,0,0,0,0,d)^2}{d^2}. \quad (123)$$

Using multiplicativity we are lead to studying

$$\sum_{M=0}^{\infty} \frac{\mu(p^M) g(1,0,0,0,0,p^M)^2}{p^{2M}} = 1 - \frac{g(1,0,0,0,0,p)^2}{p^2}. \quad (124)$$

An explicit calculation gives that

$$g(1,0,0,0,0,p) = \frac{2+p^{-1}}{1+2p^{-1}}. \quad (125)$$

Upon using this, the result in (120) follows, since

$$\frac{1}{\zeta(2)} = \prod_p (1-p^{-2})$$

and this completes the proof of Theorem 4 under the assumption of (90).

Let us write $\alpha = \alpha_1$, $\beta = \alpha_2$, $\gamma = \alpha_3$ and $\delta = \alpha_4$. Suppose that Theorem 4 is to be proved for all $|\alpha_i| \leqslant C/\log T$. Consider now (without the extra assumption (90)) any $|\alpha_i| \leqslant C/\log T$. The idea is to use Cauchy's integral formula in order to go from the previous "easier" case to the general case.

Both the LHS and the main term in the RHS of Theorem 4 are analytic functions of the complex variables α_1, α_2, α_3, α_4 (recall Remarks 10 and 12), let us denote them by $\mathrm{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $\mathrm{R}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ respectively. Let D be the polydisc defined as the Cartesian product of the open discs D_i, i.e. $D = \prod_{i=1}^{4} D_i$, where

$$D_i := \{s \in \mathbb{C} : |s - \alpha_i| < r_i\},$$

with

$$r_i = \frac{2^{i+1} C}{\log T}.$$

An application of Cauchy's integral formula yields that

$$\begin{aligned} &\mathrm{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - \mathrm{R}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \frac{1}{(2\pi i)^4} \int \cdots \int \int_{\partial D_1 \times \cdots \times \partial D_4} \frac{\mathrm{L}(\beta_1, \beta_2, \beta_3, \beta_4) - \mathrm{R}(\beta_1, \beta_2, \beta_3, \beta_4)}{(\beta_1 - \alpha_1) \cdots (\beta_4 - \alpha_4)} \, d\beta_1 ... d\beta_4. \end{aligned} \tag{126}$$

Now we notice that the β_i satisfy $|\beta_i - \alpha_i| = r_i$, which is easily seen to imply that $\beta_i \ll 1/\log T$ and that

$$|\beta_i| \geqslant 2C/\log T, \quad |\beta_i + \beta_j| \geqslant 2C/\log T \quad \text{and} \quad |\beta_i - \beta_j| \geqslant 2C/\log T.$$

This theorem thus applies if the β_i-terms are seen as shifts, so that we have

$$\mathrm{L}(\beta_1, \beta_2, \beta_3, \beta_4) - \mathrm{R}(\beta_1, \beta_2, \beta_3, \beta_4) \ll T \log^8 T. \tag{127}$$

By using (127) and considering the trivial upper bound for (126), the latter is $\ll T \log^8 T$. This finally completes the proof of Theorem 4. $\square$

4.6 Further simplification of Theorem 3 in the most standard case

Let us finish the discussion of how to evaluate the integral in (15). We take[10]

$$\{\alpha, \beta, \gamma, \delta\} = \left(\frac{i}{\log T}\right) \{\kappa + \lambda, -\lambda, -\lambda, -\kappa + \lambda\} \tag{128}$$

[10]We will later let $\lambda \to 0$.

and apply Theorem 4. One obtains an expression involving sums over m (see (85)). We now show that such sums can be dealt with by using[11]

$$\sum_{m\leqslant T^u}\frac{(T^u/m)^\alpha}{m}\approx\frac{(T^{\alpha u}-1)}{\alpha},\tag{129}$$

that is to say that after having done these replacements in all of the sums in the RHS of Theorem 4, the arising version of (85) is true.

We first prove this claim under the extra condition (90). By partial summation we have for $t\ll T$ that

$$\sum_{m\leqslant t}\frac{1}{m^{1+\alpha}}=[t]\cdot t^{-(1+\alpha)}+(1+\alpha)\int_1^t[w]\cdot w^{-2-\alpha}\,dw=\frac{(1-t^{-\alpha})}{\alpha}+O(1).\tag{130}$$

It follows immediately that

$$\sum_{m\leqslant T^u}\frac{(T^u/m)^\alpha}{m}=T^{\alpha u}\sum_{m\leqslant T^u}\frac{1}{m^{1+\alpha}}=\frac{(T^{\alpha u}-1)}{\alpha}+O(1).\tag{131}$$

Using (131) and trivial estimates, we reach our conclusion (i.e. the error parts in (131) can be absorbed into the error term in (85)).

In order to retrieve the general case from this special case, we work exactly as in the proof of Theorem 4. However, in order to apply Cauchy's integral formula we must first show analyticity of the new RHS of (85). To this end, we apply Lemma 2.5.1 in [1]. Ensuring that the conditions for applying the latter are met here essentially boils down to checking that what one gets after using[12] (129) on

$$\begin{aligned}&\sum_{m\leqslant T^u}\frac{f(x_1,x_2,T^u/m)f(x_3,x_4,T^u/m)}{m} &(132)\\ &=\sum_{m\leqslant T^u}\left\{\frac{1}{m}\Big(\frac{1}{x_1x_2}-\frac{(T^u/m)^{-x_1}}{x_1(x_2-x_1)}-\frac{(T^u/m)^{-x_2}}{x_2(x_1-x_2)}\Big)\Big(\frac{1}{x_3x_4}-\frac{(T^u/m)^{-x_3}}{x_3(x_4-x_3)}-\frac{(T^u/m)^{-x_4}}{x_4(x_3-x_4)}\Big)\right\}\end{aligned}$$

is an analytic function in terms of shifts x_1, x_2, x_3 and x_4. Explicitly one ends up with

$$\begin{aligned}&\frac{\log(T^u)}{x_1x_2x_3x_4}+\frac{(T^{-ux_3}-1)}{x_1x_2x_3^2(x_4-x_3)}+\frac{(T^{-ux_4}-1)}{x_1x_2x_4^2(x_3-x_4)}+\frac{(T^{-ux_1}-1)}{x_1^2(x_2-x_1)x_3x_4}+\frac{(T^{-ux_2}-1)}{x_2^2(x_1-x_2)x_3x_4}\\ &-\frac{(T^{-u(x_1+x_3)}-1)}{x_1x_3(x_2-x_1)(x_4-x_3)(x_1+x_3)}-\frac{(T^{-u(x_1+x_4)}-1)}{x_1x_4(x_2-x_1)(x_3-x_4)(x_1+x_4)}\\ &-\frac{(T^{-u(x_2+x_3)}-1)}{x_2x_3(x_1-x_2)(x_4-x_3)(x_2+x_3)}-\frac{(T^{-u(x_2+x_4)}-1)}{x_2x_4(x_1-x_2)(x_3-x_4)(x_2+x_4)},\end{aligned}\tag{133}$$

[11]To make sense of the RHS in the formula in the case $\alpha=0$, use the α^0-coefficient in the Taylor series.
[12]Read this as doing the relevant replacements.

which admittedly does not look too pleasant at first sight. However, there is a clever and natural strategy to employ to realise why (133) has to be analytic in x_1, x_2, x_3 and x_4.

Consider[13]

$$\Psi(x_1, x_2, x_3, x_4) := \frac{1}{(2\pi i)^2} \int_{R_2} \int_{R_1} \frac{(T^{u(s+w)} - 1)}{(s + x_1)(s + x_2)s(w + x_3)(w + x_4)w(s + w)} \, ds \, dw, \tag{134}$$

where R_1 and R_2 denote counter-clockwise rectangular paths, with vertices at $\pm 2 \pm 2i$ and $\pm 1 \pm i$ respectively. Whenever

$$x_i \neq 0 \quad \text{and} \quad x_i \neq \pm x_j \tag{135}$$

is satisfied, one notices that $\Psi(x_1, x_2, x_3, x_4)$ equals (133), by carefully applying Cauchy's Residue Theorem twice. Also, trivial estimates give that $\Psi(x_1, x_2, x_3, x_4)$ is bounded. Therefore we may conclude that all the possible singularities of (133) are removable.

In order to evaluate (15) we thus use Theorem 4 and proceed as explained above by using (129). We then substitute in (128) and view our answer as a Laurent series in terms of λ. Since the LHS of (85) remains bounded as $\lambda \to 0$, we must have cancellation so that our Laurent series actually is a Taylor series. Since we are letting $\lambda \to 0$ anyway, what all this means is that in practice one focuses term-wise on finding just the λ^0-coefficients[14].

Doing this gives us an answer in terms of κ. By seeing various symmetries in the calculations one can both save time and simplify the answer[15]. For example in the present integral-calculation one can spot that the contributions from the terms originating from the first and second term in (75) are complex conjugates[16]. This will mean that via use of Euler's formula

$$\exp(ix) = \cos x + i \sin x, \tag{136}$$

one obtains nice trigonometric terms in the answer (see (36)).

4.7 Simplified versions of Theorem 3 in two other cases

Let us recall the notation

$$M(s) = \sum_{h \leqslant T^u} \frac{1}{h^s}$$

[13] To arrive at (134), essentially one first uses Remark 11 on both the expressions in round brackets in (132), then moves the sum over m inside the double-integral and finally applies (129).

[14] Here the use of Mathematica was helpful.

[15] On a related note, by Remark 4 one could focus on finding just the analytic part of each term. Although this would save time, one advantage of keeping track of the negative κ-powers is that if all their coefficients cancel in the (total) answer, then it is "likely" that one has not made any calculation-errors!

[16] Similarly one here pairs together the terms corresponding to the third and sixth term, and the fourth and fifth term in (75).

and

$$N(s) = \sum_{h \leqslant T^u} \frac{\log(T^u/h)}{h^s}.$$

Let us also recall the notation

$$f(x_1, x_2, T_2) = \frac{1}{x_1 x_2} - \frac{T_2^{-x_1}}{x_1(x_2 - x_1)} - \frac{T_2^{-x_2}}{x_2(x_1 - x_2)}$$

and define

$$g(x_1, x_2, T_2) := \frac{\log T_2}{x_1 x_2} - \frac{1}{x_1^2 x_2} - \frac{1}{x_2^2 x_1} + \frac{T_2^{-x_1}}{x_1^2(x_2 - x_1)} + \frac{T_2^{-x_2}}{x_2^2(x_1 - x_2)}. \tag{137}$$

The following two theorems can be proved very similarly to Theorem 4.

Theorem 5. *With the same assumptions as in Theorem 4, we have*

$$\begin{aligned} &\int_{-\infty}^{\infty} M(\tfrac{1}{2} + it) N(\tfrac{1}{2} - it) \zeta(\tfrac{1}{2} + \alpha + it) \zeta(\tfrac{1}{2} + \beta + it) \zeta(\tfrac{1}{2} + \gamma - it) \zeta(\tfrac{1}{2} + \delta - it) w(t)\, dt \\ &= \Big[\int_{-\infty}^{\infty} w(t)\, dt \Big] \cdot a_3 \cdot T^{-(\alpha+\beta+\gamma+\delta)/2} \cdot \sum_{m \leqslant T^u} \frac{1}{m} \mathcal{Q}_{A,B}(T, T^u/m, g, f) + O(T \log^9 T), \end{aligned} \tag{138}$$

where we still use (86) and (87).

Theorem 6. *With the same assumptions as in Theorem 4, we have*

$$\begin{aligned} &\int_{-\infty}^{\infty} |N(\tfrac{1}{2} + it)|^2 \zeta(\tfrac{1}{2} + \alpha + it) \zeta(\tfrac{1}{2} + \beta + it) \zeta(\tfrac{1}{2} + \gamma - it) \zeta(\tfrac{1}{2} + \delta - it) w(t)\, dt \\ &= \Big[\int_{-\infty}^{\infty} w(t)\, dt \Big] \cdot a_3 \cdot T^{-(\alpha+\beta+\gamma+\delta)/2} \cdot \sum_{m \leqslant T^u} \frac{1}{m} \mathcal{Q}_{A,B}(T, T^u/m, g, g) + O(T \log^{10} T), \end{aligned} \tag{139}$$

where we again still use (86) and (87).

Similarly to how (129) was used, one may handle the above sums over m by using

$$\sum_{m \leqslant T^u} \frac{(T^u/m)^\alpha \log(T^u/m)}{m} \approx \frac{1}{\alpha^2} - \frac{T^{\alpha u}}{\alpha^2} + \frac{T^{\alpha u} \log(T^u)}{\alpha} \tag{140}$$

and

$$\sum_{m \leqslant T^u} \frac{(T^u/m)^\alpha \log^2(T^u/m)}{m} \approx -\frac{2}{\alpha^3} + \frac{2T^{\alpha u}}{\alpha^3} - \frac{2T^{\alpha u} \log(T^u)}{\alpha^2} + \frac{T^{\alpha u} \log^2(T^u)}{\alpha}, \tag{141}$$

these two formulas arising by applying partial summation to (130).

4.8 Differentiation

Some of our integrals in (15)-(24) involve differentiation. This is no problem though, as it is possible to differentiate our Theorems 4, 5 and 6 with respect to any of the shifts. To do this, we simply use "Cauchy's integral trick" and the result follows immediately. An illustration of this is now done, namely in the case when we differentiate Theorem 4 once with respect to α. The conclusion in this case is as follows:

Theorem 7. *With notation and assumptions as in Theorem 4,*

$$\begin{aligned}&\int_{-\infty}^{\infty} |M(\tfrac{1}{2}+it)|^2 \zeta'(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta+it)\zeta(\tfrac{1}{2}+\gamma-it)\zeta(\tfrac{1}{2}+\delta-it)w(t)\,dt \\ &= \Big[\int_{-\infty}^{\infty} w(t)\,dt\Big]\cdot a_3 \cdot \frac{\partial}{\partial\alpha}\Big[T^{-(\alpha+\beta+\gamma+\delta)/2}\sum_{m\leqslant T^u}\frac{1}{m}\mathcal{Q}_{A,B}(T,T^u/m,f,f)\Big] + O(T\log^9 T).\end{aligned} \tag{142}$$

Proof. Beginning with Theorem 4, we know that the LHS and the main term in the RHS of (85) are analytic functions of the complex variables α, β, γ, δ. Take the first one and subtract the latter and we get an analytic function, let us call it $D(\alpha,\beta,\gamma,\delta)$. Theorem 4 tells us that

$$D(\alpha,\beta,\gamma,\delta) \ll T\log^8 T.$$

Now use Cauchy's integral formula for the derivative with a radius $r = 1/\log T$, i.e.

$$\frac{\partial D(\alpha,\beta,\gamma,\delta)}{\partial\alpha} = \frac{1}{2\pi i}\int_{|w-\alpha|=r}\frac{D(w,\beta,\gamma,\delta)}{(w-\alpha)^2}\,dw.$$

Then the pathlength is of order $(\log T)^{-1}$ and the integrand is trivially

$$\ll (\log T)^2 \cdot T\log^8 T = T\log^{10} T,$$

giving

$$\frac{\partial D}{\partial\alpha} \ll T\log^9 T.$$

To finish the proof, remember that clearly the derivative of the difference of two analytic functions equals the difference of the derivatives of those two functions. □

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